ANALYTICAL SOLUTION OF THE ONE-DIMENSIONAL NONLINEAR INVERSE HEAT CONDUCTION PROBLEM AND ITS APPLICATIONS

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UDC 536.24:518.62

An analytical solution of the nonlinear inverse heat conduction problem is obtained and its application to identification of the boundary conditions on the components of the flow-through part of a gas turbine engine is demonstrated.

Solution of the inverse heat conduction problem (IHCP) for identification of the boundary conditions with temperature-dependent thermophysical properties (TPP) of the heat receiver material requires, as a rule, numerical methods [1]. Calculations on a computer are necessary, starting from the initial heating or cooling of the solid body in the liquid or gas flow since at that moment the temperature distribution in it is known. It should be noted that at times close to the initial moment, there is no information about changes in the temperature and its gradient at the site of temperature sensors, and therefore, to avoid using initially distorted input data, one has to manufacture thin heat receivers from materials of high thermal conductivity.

The analytical solutions of IHCP obtained for the case of constant TPP of the heat receiver material in [2] are free from this drawback. They do not require knowledge of the initial temperature distribution, and therefore the temperature and its gradient measured at any point of the body in any time interval of the process studied can be used as initial information.

I. In the present work we consider an analytical solution of the boundary-value IHCP for the practically important case where the TPP of the material of the body depend on temperature. For the overwhelming majority of heat-resistant materials used in manufacturing aircraft engines thew thermophysical properties are a linear function of the temperature T:

$$c\rho (T) = a + bT , \quad \lambda (T) = l + dT .$$

For this practically important case we have a nonlinear boundary-value IHTP for the heat conduction equation:

$$(a + bT) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left[(l + dT) \frac{\partial T}{\partial x} \right] , \quad x > 0 , \quad \tau > 0 , \tag{1}$$

with a known change of the temperature in time at the point with the coordinate $x = 0$:

$$T (0 , \tau) = g (\tau) , \quad \tau > 0 , \tag{2}$$

and with a known change of the temperature gradient at the same point:

$$\frac{\partial T (0 , \tau)}{\partial x} = \varphi (\tau) , \quad \tau > 0 . \tag{3}$$

The substitution

$$T (x , \tau) = v (x , \tau) + x\varphi (\tau) + g (\tau)$$

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 65, No. 6, pp. 735-739, December, 1993.

 1062-0125/93/6506-1240$12.50

leads to the problem with the homogeneous initial conditions

$$[a + b(v + x\varphi + g)](v_\tau + x\varphi' + g') = [l + d(v + x\varphi + g)](v_{xx} + d(v_x + \varphi)^2, \tag{1'}$$

$$v(0, \tau) = 0, \tau > 0, \tag{2'}$$

$$v_x(0, \tau) = 0, \tau > 0, \tag{3'}$$

the solution of which is sought in the form of the series

$$v(x, \tau) = \sum_{n=2}^{\infty} w_n(\tau) x^n, \tag{4}$$

which ensures satisfaction of conditions (2′) and (3′).

Substituting (4) into (1′) and equating the coefficients of the same powers of x in the left- and right-hand sides give relations for finding the functions $w_n(\tau)$:

1) at x^0

$$(a + bg) g' = 2(l + dg) w_2(\tau) + d\varphi^2, \tag{5}$$

whence $w_2(\tau)$ is obtained;

2) at x^1

$$(a + bg)\varphi' + b\varphi g' = 6(l + dg) w_3(\tau) + 6d\varphi w_2(\tau), \tag{6}$$

whence we obtain $w_3(\tau)$ with $w_2(\tau)$ already found;

3) at x^2

$$(a + bg) w_2'(\tau) + b\varphi\varphi' + bg' w_2(\tau) = 12(l + dg) w_4(\tau) + 12d\varphi w_3(\tau) + 6dw_2^2(\tau), \tag{7}$$

whence, with $w_2(\tau)$ and $w_3(\tau)$ found, we obtain $w_4(\tau)$;

4) at x^3

$$(a + bg) w_3'(\tau) + b\varphi w_2'(\tau) + b\varphi' w_2(\tau) + bg' w_3(\tau) =$$

$$= 20(l + dg) w_5(\tau) + 12d\varphi w_4(\tau) + 20dw_2(\tau) w_3(\tau) + 8d\varphi w_4(\tau), \tag{8}$$

whence, with $w_2(\tau)$, $w_3(\tau)$, and $w_4(\tau)$ known, $w_5(\tau)$ is found.

It can be shown that for $n = 4, 5, 6, \ldots$ we obtain the recurrence formula for the desired functions of time as

$$w_{n+2} = \frac{1}{(l + dg)(n + 1)(n + 2)} \Big[(a + bg) w_n' + b\varphi w_{n-1}' +$$

$$+ b\varphi' w_{n-1} + bg' w_n + b \sum_{k=2}^{n-2} w_k w_{n-k}' - d\varphi(n + 1) n w_{n+1} -$$

$$- d \sum_{k=2}^{n} w_k w_{n-k+2} k(k - 1) - 2d\varphi(n + 1) w_{n+1} -$$

$$- d \sum_{k=2}^{n} k (n - k + 2) w_k w_{n-k+2}] \cdot \tag{9}$$

Thus, the solution of nonlinear problem (1')-(3') is formally obtained as series (4), but its convergence has not been investigated yet. It can be shown that the use of formula (9) cannot give a rough estimate of $|w_n|$, $|w_n'|$, i.e., it is impossible to show that

$$|w_n| \le M, \quad |w_n'| \le M, \quad \text{where } M > 0. \tag{10}$$

Indeed, let us assume, for example, that $\varphi(\tau) = 0$, i.e., the site of measurement of the temperature $g(\tau)$ is perfectly insulated. Then, it follows from expressions (5)-(9) that $w_{2n+1} = 0$, $n = 1, 2, \ldots$. Next, let $|w_{2k}| \le M$ and $|w'_{2k}| \le M$ for $k = 1, 2, \ldots$. Then, from (9) we have the estimate

$$|w_{2n+2}| \le \frac{1}{|l + dg| (2n+1)(2n+2)} \{ M (|a + bg| + |bg'|) +$$

$$+ |b| M^2 (n - 2) + |d| M^2 [2 \cdot 1 + 4 \cdot 3 + \ldots + 2n (2n - 1)] +$$

$$+ |d| M^2 [2 \cdot 2n + 4 (2n - 2) + \ldots + 2n \cdot 2]\} = N, \tag{11}$$

If we showed that at large n the inequality $N \le M$ is satisfied, then estimate (10) would be true for any n. However, nonlinearity of the form T^2 in initial equation (1) gives a contribution to N equal to

$$Q = \frac{2 \cdot 1 + 4 \cdot 3 + \ldots + 2n (2n - 1)}{(2n + 2)(2n + 1)} >$$

$$> \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + 2n (2n - 1)}{2 (2n + 2)(2n + 1)} >$$

$$> \frac{1^2 + 2^2 + 3^2 + \ldots + (2n - 1)^2}{2 (2n + 2)(2n + 1)} = \frac{(2n - 1) 2n (4n + 1)}{2 (2n + 2)(2n + 1)} . \tag{12}$$

It follows from formula (12) that as $n \to \infty$, $N \to \infty$, too. Consequently, the estimate N in inequality (11) is not bounded absolutely by a constant, namely

$$\lim_{n \to \infty} N = + \infty . \tag{13}$$

Thus, to analyze the convergence of series (4), it seems necessary to use more precise estimates than given above. This analysis will be carried out in a separate work.

The present results are applicable to identifying the boundary conditions on the components of the flow-through part of a high-temperature gas turbine after preliminary numerical testing of the developed program. In this case, "white noise" of various levels is imposed on the functions $g(\tau)$ and $\varphi(\tau)$ found from the solution of the one-dimensional nonlinear direct unsteady-state heat-conduction problem for a plate. The thermophysical properties of the plate were approximated in the temperature range from 293 K to the melting point as follows:

a) for ZhS6 alloy

$$\lambda = 4.5 + 1.39 \cdot 10^{-2} T, \quad \text{W/(m} \cdot \text{K)},$$

$$c\rho = 2.18 \cdot 10^6 + 3.24 \cdot 10^3 T, \quad \text{J/(m}^3 \cdot \text{K)};$$

b) for ZhS6-K alloy

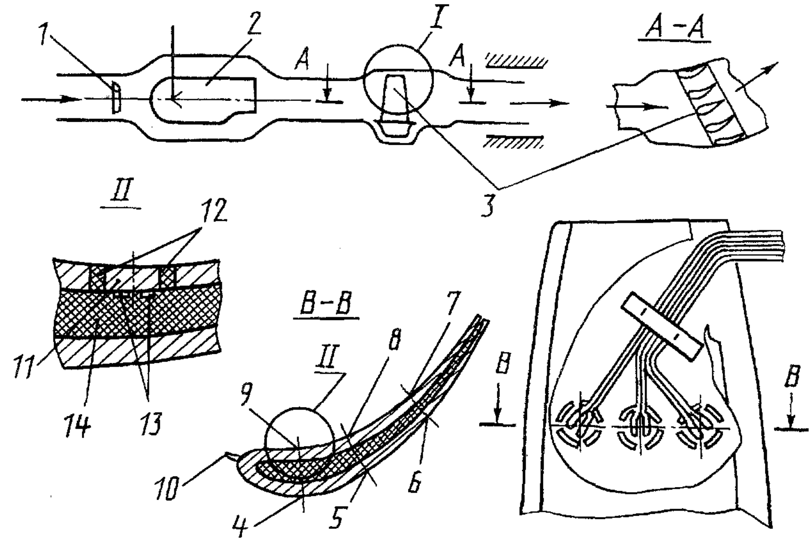

Fig. 1. Schematic of the setup for determination of the heat transfer coefficient in the turbine blade section at the side of the gas: 1) measuring section; 2) combustion chamber; 3) test blade; 4-9) points in the blade section at which the heat transfer coefficient is reconstructed; 10) thermocouple for gas temperature measurement; 11) gas receiver; 12) figure slots filled with fire cement; 13) thermal electrodes; 14) interior of the blade filled with a heat-insulating material (asbestos).

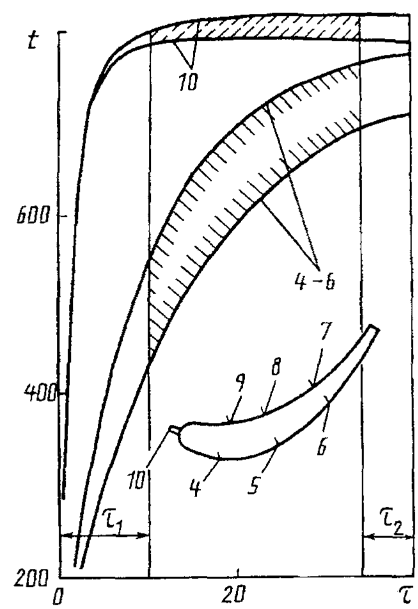

Fig. 2. Time changes of the temperatures of the gas and the inner blade wall: 4-9) temperature of the blade wall at the thermocouple sites; 10) temperature of the gas flow around the blade. t, °C; τ, sec.

$$\lambda = 2.65 + 1.96 \cdot 10^{-2}\, T,\ \ \mathrm{W/(m \cdot K)},$$

$$c\rho = 2.69 \cdot 10^{6} + 2.77 \cdot 10^{3}\, T,\ \ \mathrm{J/(m^{3} \cdot K)};$$

c) for ÉYa1T steel

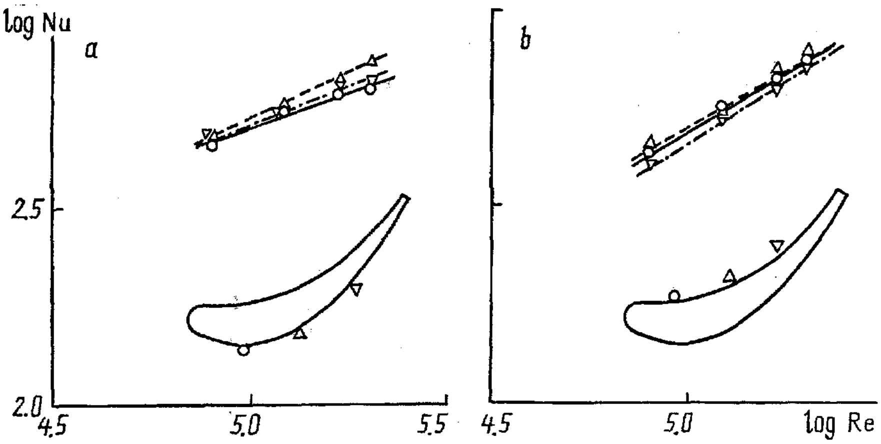

Fig. 3. Criterial relations for gas flow past the blade for the middle part of the section: a, b) for the suction face and the saddle of the blade.

$$\lambda = 10.69 + 1.57 \cdot 10^{-2}\,T, \quad \mathrm{W/(m \cdot K)},$$

$$c\rho = 3.48 \cdot 10^{6} + 0.8 \cdot 10^{3}\,T, \quad \mathrm{J/(m^{3} \cdot K)}.$$

The results of the numerical simulation have shown a high efficiency for the present method for reconstruction of the heat transfer coefficient α in its actual range on the components of the blade section in a gas turbine engine at the side of both the gas and the coolant.

II. Experiments on identifying α at the gas side were carried out on a stand whose schematic diagram is shown in Fig. 1.

Cold air, whose flow rate was determined in measuring section 1, came into combustion chamber 2 and the combustion products were blown around the preliminarily exposed blade 3 installed in a pack.

The blades were prepared for the study as follows. In two hollow blades, in middle cross sections along the fin height, windows were excised to provide access to the inner surface (in one blade at the side of the suction face and in the other, at the side of the saddle). In order to prevent electric erosion heat flow, figure slots 12 were made in the blade body, which were filled with fire cement, thus forming "one-dimensional" heat receivers 11 at points 4-9 (in Fig. 1 they are arbitrarily shown in one blade). On the surface of the heat receivers opposite the gas flow, thermal electrodes 13 were mounted; then the interior of the blades 14 was filled with asbestos and the excised parts of the suction face and the saddle were returned to their places.

The flow rate of the cold air in the test was 0.25 to 0.65 kg/sec for one blade passage; this rate corresponded to the range Re = $(0.82\text{-}2.20) \cdot 10^{5}$ of Reynolds numbers for the gas constructed with the chord length as the characteristic dimension.

In Fig. 2 one can see the ranges and time changes of temperature both in the gas flow (it was measured by thermocouple 10, fixed on the leading edge of the blade) and at points 4-9 on the inner surface of the blade.

It should be noted that in the tests each run with a particular gas flow rate was repeated three times, and therefore the reliability of the functions $g(\tau)$ smoothed then by cubic splines at $\varphi(\tau) = 0$ increased. In order to increase the smoothing accuracy, a definite part of the temperature range was chosen: we rejected the initial time interval of length τ_1 as undeveloped in the gasdynamic respect and the final time interval τ_2, starting from the time that the temperature difference between the sensor and the medium became commensurable with the instrumental error. Then, solution (4), in which the first three nonzero terms of the series were kept, was used to reconstruct the time functions of temperature and its gradient at points 4-9, and, consequently, the heat transfer coefficients α were also reconstructed at the measured temperature.

The results of the tests carried out with the present methods give stable values of α. Dimensionless values of α at the gas side are shown in Fig. 3 as the dimensionless function $\log \mathrm{Nu} = f(\log \mathrm{Re})$ at the characteristic points of the blade section of a gas turbine engine.

NOTATION

$T(x, \tau)$, temperature; x, coordinate; τ, time; $g(\tau)$ and $\varphi(\tau)$, temperature and its gradient at the temperature sensor site; c, specific heat; ρ, density; λ, thermal conductivity; α, heat transfer coefficient; Nu, Nusselt number; Re, Reynolds number.

REFERENCES

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